Def Let X be a complex manifold. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called locally free of rank r if for any $x \in X$ it there exists $p \in \mathcal{U} \subseteq X$ such that

$$\mathcal{F}_{|\mathcal{U}} \cong_{\phi_{\mathcal{U}}} \bigoplus_{i=1}^{r} \mathcal{O}_{X|\mathcal{U}} \quad \text{as sheaves on } \mathcal{U}.$$

§3. Vector bundles

Def A (holomorphic) vector bundle E of X (connected) is a collection of complex vect. spaces $\{E_x\}_{x\in X}$ s.t. $E = \bigcup_x E_x$ has a structure of complex manifold, the projection map

$$\pi : E \to X$$

is holomorphic, and for any $x \in X$ it there exists $\mathcal{U} \ni p$ and a biholomorphism

$$\phi_{\mathcal{U}} : \bigcup_{x\in\mathcal{U}} E_x \longrightarrow \mathcal{U} \times \mathbb{C}^r$$

such that $E_x \xrightarrow{\phi_{\{x\}}} \{x\} \times \mathbb{C}^r$ is an isomorphism.

- $\phi_{\mathcal{U}}$ is called trivialization of E.
- the number r is independent by the trivialization and it is called RANK of E.

Remark For any $x \in \mathcal{U} \cap V$ we have

$$\phi_U \circ \phi_V^{-1} : \{x\} \times \mathbb{C}^r \longrightarrow \{x\} \times \mathbb{C}^r$$
$$(x, v) \longmapsto (x, g_{UV}(x) v)$$

with $g_{UV}(x) \in GL(r, \mathbb{C})$.

Thus we have functions

$$g_{UV} : U \cap V \to GL(r, \mathbb{C})$$

that are entry-wise holomorphic and that completely determine the trivializations $\phi_U \circ \phi_V^{-1}$.

They satisfy the relation

$$\begin{cases} g_{UV} = g_{VU}^{-1} \\ g_{UV} \circ g_{VW} \circ g_{WU} = Id_{\mathbb{C}^r} \end{cases} \qquad (*)$$

Def

The collections $\{g_{UV}\}_{UV}$ are called transit. functs of the vector bundle E.

Def

We say that E and F on X are isomorphic if it there exists $f: E \to F$ bihotom. s.t.

$\pi_F \circ f = \pi_E$ (i.e. $\begin{array}{ccc} E & \xrightarrow{f} & F \\ & {\scriptstyle \pi_E} \searrow \; X \; \swarrow {\scriptstyle \pi_F} & \end{array}$) and

$f_x : E_x \to F_x$ is an isomorphism.

Remark

Notice that if $\{g_{UV}\}$ and $\{h_{UV}\}$ are cocycles of E and F, then

$$h_{UV} = (\Psi_U \circ \Psi_V^{-1}) = (\Psi_U \circ f \circ \phi_U^{-1}) \circ g_{UV} \circ \underbrace{(\phi_V \circ f^{-1} \circ \Psi_V^{-1})}_{(\Psi_V \circ f \circ \phi_V)^{-1}}$$

on $U \cap V$.

Thus, there are two local biholomorphisms $f_{\mathcal{U}} \in GL(r, \mathcal{O}(U))$, $f_V \in GL(r, \mathcal{O}(V))$ such that $h_{UV} = f_{\mathcal{U}} \circ g_{UV} \circ f_V^{-1}$.

Remark 2

Let us consider a complex (connected) mfd X and assume to have on a open cover $\underline{\mathcal{U}} = \{U_\alpha\}_\alpha$ a set of holom. fcts $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})\}_{\alpha\beta}$ s.t. (*) holds. Then we define

$$E := \bigcup_\alpha U_\alpha \times \mathbb{C}^r$$

where we identify two points

$$(p, v) \in U_\alpha \times \mathbb{C}^r, \quad (q, w) \in U_\beta \times \mathbb{C}^r \overset{\text{def}}{\iff}$$

$$q = p \quad \text{and} \quad w = g_{\beta\alpha}(p) \cdot v.$$

The set E has a natural complex structure given by the inclusions $U_\alpha \times \mathbb{C}^r \hookrightarrow E$, so that together with the map $\pi : E \to X$, $(p, v) \mapsto p$ it is a rank r vector bundle on X, with triv. maps

$$\varphi_\alpha : \bigcup_{x \in U_\alpha} E_x \longrightarrow U_\alpha \times \mathbb{C}^r$$
$$(p, \lambda_1, \dots \lambda_r) \longmapsto \left(p, g_{\alpha\gamma}(p) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{pmatrix}\right) \text{ if } p \in U_\alpha \cap U_\gamma$$

Furthermore, if E is a vect. bundle on X with cocycles $\{g_{\alpha\beta}\}_{\alpha\beta}$ and F is the vect. bundle constructed by $\{g_{\alpha\beta}\}$, then $E \cong F$.

We obtained that the set of rank r vector bundles on X trivialized by an open cover $\underline{\mathcal{U}}=\{U_\alpha\}_\alpha$ can be identified with the set

(namely, it satisfies (*) properties)

$$\{(g_{\alpha\beta})_{\alpha\beta} \mid g_{\alpha\beta} \text{ is a cocycle}\} \Big/ \sim$$

where $\sim$ is the relation

$$g_{\alpha\beta} \sim h_{\alpha\beta} \iff \exists \{f_\alpha \in GL(r, \mathcal{O}(U_\alpha))\}_\alpha \text{ s.t.}$$

$$h_{\alpha\beta} = f_\alpha \cdot g_{\alpha\beta} \cdot f_\beta^{-1}$$

Remark

We observe that

$$Z^1(\underline{\mathcal{U}}, GL(r,\mathcal{O})) = \{(g_{\alpha\beta})_{\alpha\beta} \mid g_{\alpha\beta} \text{ is a cocycle}\}$$

because the property to be a cocycle in the sense of (*) is equivalent to $\delta g_{\alpha\beta} = 0$, where δ is the coboundary operator.

Assume that $\boxed{r=1}$. Then $\sim$ translates as

$$g_{\alpha\beta} \sim h_{\alpha\beta} \iff \exists \{f_\alpha \in \mathcal{O}^*(U_\alpha)\}_\alpha \text{ s.t.}$$

$$h_{\alpha\beta} = f_\alpha \cdot g_{\alpha\beta} f_\beta^{-1} = (\delta f^{-1})_{\alpha\beta}\, g_{\alpha\beta}$$

commutativity of $\mathcal{O}^*(U_\alpha \cap U_\beta)$

This means that $\sim$ is the coboundary relation and the above quotient is $H^1(\underline{\mathcal{U}}, \mathcal{O}^*)$.

What does it happen for $\boxed{r>1}$?
In this case we are working with the sheaf

$$GL(r, \mathcal{O})$$

which is NOT a sheaf of abelian groups.
In this case the coboundary operator $\delta: C^p \to C^{p+1}$ is still well defined with the property that

$$\delta^2 = 0 \text{ and so } \delta(C^p) \subseteq Z^{p+1}.$$

However,

1. δ is $\boxed{\text{NOT}}$ an homomorphism;
2. Z^{p+1} is $\boxed{\text{NOT}}$ a group;
3. $\delta(C^p) \subseteq Z^{p+1}$ is $\boxed{\text{NOT}}$ a group;

In this case it is natural to construct a cohomology for which $H^1(\underline{\mathcal{U}}, GL(r, \mathcal{O}))$ still remains in bijection with the above quotient, although we could lose the structure of group:

Non-abelian 1^{st} Čech cohomology:

called Gauge relation

Given $\sigma, \tau \in Z^1(\underline{\mathcal{U}}, \mathcal{F})$, then $\sigma \sim \tau \overset{\text{def}}{\Longleftrightarrow} \exists f \in C^0(\underline{\mathcal{U}}, \mathcal{F})$ s.t.

$$\tau_{\alpha\beta} = (f_\alpha|_{U_\alpha \cap U_\beta}) \cdot \sigma_{\alpha\beta} \cdot (f_\beta|_{U_\alpha \cap U_\beta})^{-1}.$$

$$H^1(\underline{\mathcal{U}}, \mathcal{F}) := Z^1(\underline{\mathcal{U}}, \mathcal{F}) / \sim_{gauge}$$

Thm The set of rank r vector bundles on X trivialized by $\underline{\mathcal{U}} = \{U_\alpha\}_\alpha$ is in corresp. with

$$H^1(\underline{\mathcal{U}}, GL(r, \mathcal{O}))$$

In general, the set of rank r vector bundles of X is in corresp. with

$$\boxed{H^1(X, GL(r, \mathcal{O}))}$$

Def $H^1(X, \mathcal{O}^*)$ is called PICARD GROUP of X, and it is denoted by $\mathrm{Pic}(X)$.

Since $\mathcal{O}^*$ is commutative, $\mathrm{Pic}(X)$ is an abelian group and it encodes the line bundles of X (:= rank 1 vect. bundles), by the previous theorem.

Examples 1) $X \times \mathbb{C}^r$ is the trivial vect. bundle of rank r, its cocycles are $\{Id_{\mathbb{C}^r}\}_\alpha$

A vector bundle is then trivial $\Leftrightarrow$ its cocycles $\{g_{\alpha\beta}\}_{\alpha\beta}$ are $g_{\alpha\beta} = f_\alpha f_\beta^{-1}$ on $U_\alpha \cap U_\beta$, and $f_\alpha \in GL(r, \mathcal{O}(U_\alpha))$

2) $X = \mathbb{P}^n$, we have a natural vector bundle, called Tautological Bundle:

$$E := \{[\bar{v}, w] \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid w \in \langle v \rangle\} \xrightarrow{\pi} \mathbb{P}^n$$
$$[\bar{v}, w] \longmapsto \bar{v}$$

Given $U_{x_i} = \{x_i \neq 0\}$, then we have trivializations

$$\pi^{-1}(U_{x_i}) \xrightarrow{\varphi_i} U_{x_i} \times \mathbb{C}$$
$$[\bar{v}, w] \longmapsto (\bar{v}, w_i)$$
$$[\bar{v}, \tfrac{\lambda}{v_i} v] \longleftarrow (\bar{v}, \lambda)$$

The cocycles are then

$$g_{ji} = \varphi_j \circ \varphi_i^{-1} : U_{x_i} \cap U_{x_j} \longrightarrow \mathbb{C}^*$$
$$[x_0, \dots x_n] \longmapsto \frac{x_j}{x_i}$$

$\Rightarrow \left\{ g_{ji} = \frac{x_j}{x_i} \right\}_{i=0,\dots n}$ define the tautological bundle of $\mathbb{P}^n$, which is a line bundle.

3) Any operation among vector spaces induces new vector bundles;

Given a rank r vector bundle $E \xrightarrow{\pi} X$ with cocycles $\{g_{\alpha\beta}\}_{\alpha\beta}$, and a rank s vect. bundle $F \xrightarrow{\pi} X$ with cocycles $\{h_{\alpha\beta}\}_{\alpha\beta}$, then

(Dual) $E^* \to X$ is the vect. bundel of rank r defined by $\{(g_{\alpha\beta}^t)^{-1}\}_{\alpha\beta}$;

(Direct Sum) $E \oplus F \to X$ is the vect. bundle or rank $r+s$ given by $\left\{\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta}\end{pmatrix}\right\}_{\alpha\beta}$

(Tensor product) $E \otimes F \to X$ is the vect. bundle of rank $r \cdot s$ defined by
$$\{g_{\alpha\beta} \otimes h_{\alpha\beta}\}_{\alpha\beta}$$
with $g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in GL(n, \mathbb{C}^r \otimes \mathbb{C}^s)$

Remark If both E and F are line bundle, then $E \otimes F$ is still a line bundle, so we have a natural operation on the sets of line bundles of X. This operation makes the set a group and the previous bijection
$$(\{\text{line bundles}\}, \otimes) \xrightarrow{\sim} Pic(X)$$
becomes an isomorphism of groups.

(Alt. power) $\Lambda^k E \xrightarrow{\pi} X$ is the vector bundle of rank $\binom{n}{k}$ defined by

$$\{\Lambda^k g_{\alpha\beta}\}_{\alpha\beta}$$

where $\Lambda^k g_{\alpha\beta}(x) \in GL(\Lambda^k \mathbb{C}^n)$

(Determinant) $\Lambda^n E \longrightarrow X$ is always a line bundle, with cocycles

denoted by $\det(E)$

$$\{\det(g_{\alpha\beta})\}_{\alpha\beta}$$

This is called determinant bundle.

Remark On X we always have two natural rank n vector bundles, where $n := \dim(X)$:

$$TX \longrightarrow X \qquad\qquad T^*X \longrightarrow X$$

with cocycles (transit. functions of X) $\{J(\varphi_\alpha \circ \varphi_\beta^{-1})\}_{\alpha\beta}$ with cocycles $\{[J^T(\varphi_\alpha \circ \varphi_\beta^{-1})]^{-1}\}_{\alpha\beta}$

Then we always have a natural line bundle

$$\det(T^*X) \longrightarrow X$$
$$(\overset{\shortparallel}{\Lambda^n T^*X})$$

with cocycles $\{\det J^{-1}(\varphi_\alpha \circ \varphi_\beta^{-1})\}_{\alpha\beta}$.

Def $\det(T^*X)$ is called CANONICAL BUNDLE of X, and it is denoted by ω_X.

(Pullback) Let $E \xrightarrow{\pi} Y$ be a vect. bundle and $X \xrightarrow{f} Y$ be a holomorphic map. Then the pullback bundle is a vector bundle $f^*E \to X$ defined as

$$\underset{\{(p,e)\in X\times E \mid \pi(e)=f(p)\}}{\overset{\|}{\;}}$$

$$\begin{tikzcd} E\times_Y X \arrow[r] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

This is also known as fibre product of π and f.

The trivializating maps are given in this way:

let $p\in X$, then $f(p)\in Y$ and it there exists $\mathcal{U}\ni f(p)$ and a biholomorphic map

$$\varphi_{\mathcal{U}}: \pi^{-1}(\mathcal{U}) \to \mathcal{U}\times\mathbb{C}^r$$

But then we define the trivialization

$$\begin{aligned}\Psi_{\mathcal{U}}: (\pi')^{-1}(f^{-1}(\mathcal{U})) &\longrightarrow f^{-1}(\mathcal{U})\times\mathbb{C}^r\\ (q,e) &\longmapsto (q, \varphi_{\mathcal{U}}(e))\end{aligned}$$

and the cocycles are

$$\{g_{V\mathcal{U}}\circ f : f^{-1}(\mathcal{U}\cap V) \to GL(r,\mathbb{C})\}_{\mathcal{U}V}$$

Thus, if $\{U_\alpha\}_\alpha$ is a trivializating open cover of $E \xrightarrow{\pi} Y$ with cocycles $\{g_{\alpha\beta}\}_{\alpha\beta}$, then $\{f^{-1}(U_\alpha\cap U_\beta)\}_{\alpha\beta}$ is a trivializ. open cover of $f^*E \xrightarrow{\pi'} X$ with cocycles

$$\{g_{\alpha\beta}\circ f : f^{-1}(U_\alpha\cap U_\beta) \to GL(r,\mathbb{C})\}_{\alpha\beta}$$

§3.1 Correspondence Vector bundles – Locally free sheaves

Def Given $\mathcal{U} \subseteq X$, a section of $E \xrightarrow{\pi} X$ over $\mathcal{U}$ is a holom. map

$$\sigma : \mathcal{U} \to E$$

such that $\pi \circ \sigma = \mathrm{Id}_{\mathcal{U}}$.

We can easily define the sum two sections over $\mathcal{U}$ and the product of a holomorphic function f on $\mathcal{U}$ and a section over $\mathcal{U}$.
Thus, $\Gamma(\mathcal{U}, E) := \{\text{sections over } \mathcal{U} \text{ of } E \xrightarrow{\pi} X\}$ is a sheaf of X of $\mathcal{O}_X$-modules.

Def A *frame* of $E \xrightarrow{\pi} X$ over $\mathcal{U}$ is a collection of $\sigma_1, \dots \sigma_r$ sections over $\mathcal{U}$ s.t.

$\sigma_1(x), \dots \sigma_r(x)$ are lin. indip. $\forall x \in \mathcal{U}$.
(and so a basis of $\mathbb{C}^r$)

Remark A natural frame of $E \xrightarrow{\pi} X$ over a trivializing open set $\mathcal{U} \subseteq X$ of $E \xrightarrow{\pi} X$ with trivialization $\varphi_{\mathcal{U}} : \bigcup_{x \in \mathcal{U}} E_x \to \mathcal{U} \times \mathbb{C}^r$ is the following:
$e_1, \dots e_r$ stand. basis of $\mathbb{C}^n$

$$\underbrace{\varphi_{\mathcal{U}}^{-1}(x, e_1)}_{\sigma_1(x)}, \dots\dots, \underbrace{\varphi_{\mathcal{U}}^{-1}(x, e_r)}_{\sigma_r(x)}$$

Given a section $\sigma : V \subseteq \mathcal{U} \to E$, then σ can be written as $\varphi_{\mathcal{U}} \circ \sigma(x) = (f_1(x), \dots f_r(x))$, so

$$\sigma(x) = \sum_{i=1}^{r} f_i(x) \cdot \sigma_i(x) \quad \text{with } f_i \in \mathcal{O}(V).$$

This defines a natural isomorphism $\mathcal{O}_X$-modules

$$\Gamma(\mathcal{U}, E)_{|V} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_X(\mathcal{U})_{|V}$$
$$\sigma \longmapsto (f_1, \dots f_r)$$

We have proved that $\Gamma(-, E)$ is a locally free sheaf of rank r.

Conversely, let us consider a locall free sheaf $\mathcal{F}$ of rank r. Then if there exists an open cover $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}_\alpha$ s.t. $\mathcal{F}|_{\mathcal{U}_\alpha} \underset{\sim}{\xrightarrow{\phi_\alpha}} \bigoplus_{i=1}^r \mathcal{O}_X|_{\mathcal{U}_\alpha}$

Thus, we have maps:

$$\underbrace{\bigoplus_{i=1}^r \mathcal{O}_X(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)}_{\mathcal{O}_X|_{\mathcal{U}_\alpha}(\mathcal{U}_\beta)} \xrightarrow{\phi_\alpha^{-1}} \mathcal{F}|_{\mathcal{U}_\alpha}(\mathcal{U}_\beta) = \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = \mathcal{F}|_{\mathcal{U}_\beta}(\mathcal{U}_\alpha) \xrightarrow{\phi_\beta} \bigoplus_{i=1}^r \underbrace{\mathcal{O}_X(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)}_{\mathcal{O}_X|_{\mathcal{U}_\beta}(\mathcal{U}_\alpha)}$$

that means

$$\bigoplus_{i=1}^r \mathcal{O}_X(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \xrightarrow{\overbrace{\phi_\beta \circ \phi_\alpha^{-1}}^{\phi_{\beta\alpha}}} \bigoplus_{i=1}^r \mathcal{O}_X(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is an isomorphism of $\mathcal{O}_X(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$-modules.

But then given $(f_1, \dots f_r) \in \bigoplus_{i=1}^r \mathcal{O}_X(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, we have

$$\phi_{\beta\alpha}(f_1, \dots f_r) = \phi_{\beta\alpha}\left(\sum f_j e_j\right) = \sum f_j \phi_{\beta\alpha}(e_j)$$

so $(f_1, \dots f_r) \xrightarrow{\phi_{\beta\alpha}} (\phi_{\beta\alpha}(e_i))_i \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$

columns of the matrix

However, $\phi_{\beta\alpha}$ is an iso, so $g_{\beta\alpha} := (\phi_{\beta\alpha}(e_i))_i$ is an invertible matrix on $\mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$.

By construction, it is clear that

- $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ on $\mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$.
- $g_{\beta\gamma} \circ g_{\gamma\alpha} \circ g_{\alpha\beta} = \mathrm{Id}$ on $\mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma)$.

We have constructed cocycles of X $\{g_{\beta\alpha}\}_{\beta\alpha}$ which define a vector bundle $E \xrightarrow{\pi} X$ with transition functions

$E = \bigsqcup_\alpha \mathcal{U}_\alpha \times \mathbb{C}^r / \sim$

$$\Psi_\alpha : \bigcup_{x\in\mathcal{U}_\alpha} E_x \longrightarrow \mathcal{U}_\alpha \times \mathbb{C}^r$$
$$(p, \lambda_1, \dots \lambda_r) \longmapsto \left(p, g_{\alpha\gamma}(p)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_r\end{pmatrix}\right) \text{ if } p\in\mathcal{U}_\alpha\cap\mathcal{U}_\gamma$$

Furthermore, we have an iso given $\mathcal{U}\subseteq V$ as follows

$$\begin{aligned} \mathcal{F}(\mathcal{U}) &\longrightarrow \Gamma(\mathcal{U}, E) \\ s &\longmapsto s: \mathcal{U} \longrightarrow E \\ \| \quad & \qquad\quad x \longmapsto [(x, s_\alpha(x))] \\ (s_\alpha)_\alpha & \end{aligned}$$

$\cap$

$\mathcal{F}_{|\mathcal{U}_\alpha}(\mathcal{U}) \underset{\phi_\alpha}{\cong} \bigoplus \mathcal{O}_{X|\mathcal{U}_\alpha}(\mathcal{U})$ and $s_\beta = g_{\beta\alpha} s_\alpha$

where $g_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$

Thus, $\mathcal{F} \longrightarrow \Gamma(-, E)$ is an ISOMORPHISM

We have proved the correspondence

$$\boxed{\begin{array}{c} \{\text{Locally free sheaves of } X \text{ of rank } r\}/\text{iso} \\ \updownarrow \\ \{\text{rank } r \text{ vect. bundles on } X\}/\text{iso} \\ \updownarrow \\ H^1(X, GL(r, \mathcal{O})) \end{array}}$$

which become isomorphisms of groups for $\boxed{r=1}$

line bundles

Remark Given $f: X \to Y$, then we have the following maps:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Locally free sheaves of } Y \\ \text{of rank } r\end{array}\right\}\Big/_{\text{iso}} \arrow[r, "f^*"] \arrow[d, leftrightarrow] & \left\{\begin{array}{c}\text{Locally free sheaves of } X \\ \text{of rank } r\end{array}\right\}\Big/_{\text{iso}} \arrow[d, leftrightarrow] \\
\left\{\begin{array}{c}\text{rank } r \text{ vect.} \\ \text{bundles on } Y\end{array}\right\}\Big/_{\text{iso}} \arrow[r, "f^*"] \arrow[d, leftrightarrow] & \left\{\begin{array}{c}\text{rank } r \text{ vect.} \\ \text{bundles on } X\end{array}\right\}\Big/_{\text{iso}} \arrow[d, leftrightarrow] \\
H^1(Y, GL(r, \mathcal{O})) \arrow[r, "f^*"] & H^1(X, GL(r, \mathcal{O}))
\end{tikzcd}$$

$$\mathcal{F} \longmapsto f^*\mathcal{F} \quad \text{(inverse image sheaf)}$$

$$(E \to Y) \longmapsto f^*E \to X \quad \text{(pullback bundle)}$$

$$\{g_{\alpha\beta}\}_{\alpha\beta} \longmapsto \{g_{\alpha\beta} \circ f\}_{\alpha\beta}$$

and when $\boxed{r=1}$ then the horizontal map

line bundles

are homomorphisms of groups.

§4 Divisors

Let X be a complex manifold of dim. n. (Here we don't need X to be compact).

An analytic hypersurface $V \subseteq X$ is an analytic subvariety of dimension $n-1$.

Fact 0 V is defined locally around a point $p \in V$ by a single holomorphic function f.

Warning! Fact 0 holds because X is smooth. If we want to work with something singular, we need to distinguish between Cartier and Weil divisors.

Fact 1 Furthermore, if g is a holomorphic function defined locally at p and vanishing on V, then g is divisible by f in a neighborhood of p by a function not vanishing at p.

Thus, f is called local defining function of V at p.

Fact 2 Any analytic hypersurface V can be decomposed as a finite union of irreducible analyt. hypersurfaces

$$V = V_1 \cup \dots \cup V_m.$$

Def A (Cartier) divisor D on X is a formal
"$\forall p \in X$ $\exists U \ni p$ that intersects only a finite number of V_i"
locally finite integer combination of irr. analytic hypersurfaces:

$$D = \sum a_i V_i \quad , \quad a_i \in \mathbb{Z}.$$

Div(X) is the set of Cartier divisors of X, which is a group with the natural sum.

- D is effective if $a_i \geq 0$ for each i.

Let V be an irreducible hypersurf. of X, $p \in V$, and g be a local holomorphic fuction near p.
Then, if f is a local defin. function of V, from Fact 1 g is divided by f:

$$g = h f^a \quad , \quad h(p) \neq 0$$

We define $\operatorname{ord}_{V,p}(g) := a$.

Fact 3: $\operatorname{ord}_{V,p}(g)$ is indipendent by the choice of $p \in V$. Thus, we can define

$$\operatorname{ord}_V(g) := \operatorname{ord}_{V,p}(g).$$

Remark $\mathrm{ord}_V(gh) = \mathrm{ord}_V(g) + \mathrm{ord}_V(h)$

Let f be a meromorphic function on $p \in U \subseteq X$, written locally as $f = \frac{g}{h}$ around $p \in V$. Then we define

$$\mathrm{ord}_V(f) := \mathrm{ord}_V(g) - \mathrm{ord}_V(h)$$

Given a global meromorphic function f on X, then we may associate to f the divisor

$$\mathrm{div}(f) := \sum_V \mathrm{ord}_V(f) \cdot V$$

Def $\mathrm{div}(f)$ is called Principal Divisor

The set of principal divisors is a subgroup of $\mathrm{Div}(X)$ and is denoted by

$$\mathrm{PDiv}(X) \leq \mathrm{Div}(X)$$

Example $X = \mathbb{P}^2$, $f = \{(X, \frac{x_1}{x_0})\}$ is a global merom. function of X $\Rightarrow$ $\mathrm{div}(f) = (x_1) - (x_0) = \{x_1 = 0\} - \{x_0 = 0\}$

Instead, $f = \{(X, \frac{x_1^2}{x_0})\}$ have $\mathrm{div}(f) = 2(x_1) - 2(x_0)$.